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university of
groningen

Tentamen Introduction to Plasma Physics

June 23, 2015, 9:00 - 12:00 h

You may use one sheet of notes and a (graphical) calculator. A sheet with plasma formulae is attached. **Please motivate all your answers**, an answer without motivation will not be considered. You can use either english or dutch. Success !

Please write on every sheet your name and student number.

Free: $\boxed{10}$

1. Consider a plasma consisting of electrons and ions of equal number densities, i.e. $n = n_e = n_i$, but the electrons and ions have different temperatures T_e and T_i, respectively. Assume that the electron density is proportional to $\exp(e\Phi/k_BT_e)$ and that the ion density is proportional to $\exp(-e\Phi/k_BT_i)$.

   (a) $\boxed{4}$ Explain qualitatively the concept of Debye length.

   (b) $\boxed{8}$ Show that the Debye length, λ_D, is given by

   $$\frac{1}{\lambda_D^2} = \frac{1}{\lambda_{De}^2} + \frac{1}{\lambda_{Di}^2} .$$

   with λ_{De} and λ_{Di} the Debye lengths of the electrons and ions, respectively.

   (c) $\boxed{4}$ Which assumption(s) did you make in the derivation of λ_D?

   (d) $\boxed{4}$ How do the electron and ion Debye lengths compare if $T_e = T_i$?

2. A plasma is trapped in an open min-B trap with magnetic field $\mathbf{B}_0(s)$, with s the distance along a magnetic field line. There is also a parallel electric field that can be described by a monotonic electrostatic potential $\Phi(s)$. Suppose that an absorbing boundary (e.g. the wall of the plasma chamber) exists at $s = m$.

   (a) $\boxed{4}$ Give at least two possible mechanisms of particle losses in open min-B traps.

   (b) $\boxed{7}$ By assuming conservation of energy and magnetic moment show that, at an arbitrary point s, the loss cone in velocity space $(v_\parallel, v_\perp)$ is given by

   $$v_\parallel^2 - \underbrace{\left(\frac{B_m}{B} - 1\right)}_{>0} v_\perp^2 = \frac{2q}{m}\underbrace{(\Phi_m - \Phi)}_{>0} .$$

   with B_m the magnetic induction and Φ_m the electrostatic potential at $s = m$ and q and m the charge and mass of the trapped particle, respectively.

   (c) $\boxed{7}$ Sketch the shape of the loss cone for electrons and ions assuming that $\Phi_m > \Phi$ and $B_m > B$.

   (d) $\boxed{7}$ Suppose ions are now emitted at $s = m$. What region of $(v_\parallel, v_\perp)$ is accessible to the ions again assuming that $\Phi_m > \Phi$ and $B_m > B$.

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3. The MHD induction equation is given by

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B} \,.$$

(a) $\boxed{5}$ Derive this equation and give a physical interpretation of the two terms on its right-hand side.

(b) $\boxed{5}$ Show that the following function

$$\mathbf{B} = B_0 \,\mathrm{erf}\left[\frac{1}{2}\left(\frac{\mu_0 \sigma}{t}\right)^{1/2} y\right] \mathbf{e}_x$$

is a solution of the flow-free ($\mathbf{u} = 0$) MHD induction equation with the error function $\mathrm{erf}(y)$ defined as

$$\mathrm{erf}(y) = \frac{1}{\sqrt{\pi}} \int_0^y e^{-v^2} dv \,.$$

(c) $\boxed{5}$ Make a plot of $B_x(y, t)$ as a function of y for a series of times $t_1 < t_2 < t_3$.

(d) $\boxed{5}$ Compute the current density $j_z(y, t)$ and plot it as a function of y for a series of times $t_1 < t_2 < t_3$.

(e) $\boxed{5}$ Assuming pressure balance, $B^2/(2\mu_0) + P = \text{constant}$, make a plot of the pressure function $P(y)$ that would be required to achieve an equilibrium state for this configuration.

4. Consider a streaming instability in a one-dimensional collisionless plasma in which the electrons with mass m_e move with velocity v_0 through a neutralizing background of stationary ions with mass m_i. There is no magnetic field and both electrons and ions are cold, i.e. $T_e = T_i = 0$. Assume that the zero-order normalized distribution functions of the electrons and ions are $F_{e0} = \delta(v - v_0)$ and $F_{i0} = \delta(v)$, respectively.

(a) $\boxed{5}$ Give the dielectric function of the plasma for small-amplitude, steady-state electrostatic waves taking both electrons and ions into account (see formula sheet).

(b) $\boxed{7}$ Derive the following dispersion relation:

$$\frac{m_e}{m_i} \frac{\omega_{pe}^2}{\omega^2} + \frac{\omega_{pe}^2}{(\omega - k v_0)^2} = 1 \,.$$

(c) $\boxed{5}$ Use a diagram to show how the dispersion relation can be solved graphically and discuss the (in)stability of this electrostatic mode.

(d) $\boxed{3}$ Is it possible to derive the dispersion relation also using the MHD model? Explain your answer.

Total: $\boxed{100}$

Formula sheet

Vector double cross product:
$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Integrals: $\int_{\infty}^{\infty} e^{-x^2} \mathrm{d}x = \sqrt{\pi}$

Characteristic plasma parameters:
$\lambda_{dB} = \sqrt{\frac{\epsilon_o k_B T_e}{n_e e^2}} = 743 \sqrt{\frac{T_e(\mathrm{eV})}{n_e(\mathrm{cm}^{-3})}}\,(\mathrm{cm}) \quad ; \quad \nu_{pe} = \frac{\omega_{p,e}}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{e^2 n_e}{m_e \epsilon_o}} = 8.98\sqrt{n_e(\mathrm{cm}^{-3})}\,(\mathrm{kHz})$
$\nu_{ce}(\mathrm{GHz}) = 28\,B(\mathrm{T}) \quad ; \quad \nu_{ee}(\mathrm{s}^{-1}) = 4 \cdot 10^{-12} \frac{n_e(\mathrm{m}^{-3}) \ln\Lambda}{T_e^{3/2}(\mathrm{eV})}$

Guiding center drifts:
$\mathbf{v}_{gc,E\times B} = \frac{\mathbf{E}\times\mathbf{B}}{B^2} \quad ; \quad \mathbf{v}_{gc,\nabla B} = \frac{mv_\perp^2}{2qB^3}\mathbf{B} \times \nabla B \quad ; \quad \mathbf{v}_{gc,cB} = \frac{mv_\parallel^2}{qB^3}\mathbf{B}\times\nabla B$

Adiabatic invariants:
$J_1 = \frac{m}{q}\mu_L = \frac{m}{q}\frac{\frac{1}{2}mv_\perp^2}{B} \quad ; \quad J_2 = m\langle v_\parallel \rangle L/\pi \quad ; \quad J_3 = \frac{q\Phi}{2\pi}$

Electro- and magnetostatics:
$\epsilon_0 \oint_S \mathbf{E} \cdot \mathbf{n}\,\mathrm{d}\sigma = Q_{encl} \quad ; \quad \oint_C \mathbf{B} \cdot \mathbf{t}\,\mathrm{d}s = \mu_0 I_{encl}$
$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$

Maxwell's equations:
$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad ; \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \epsilon_0\mu_0 \frac{\partial \mathbf{E}}{\partial t} \quad ; \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad ; \quad \nabla \cdot \mathbf{B} = 0$
with $\rho = \sum_\alpha q_\alpha \int f_\alpha(\mathbf{r}, \mathbf{v}, t)\,\mathrm{d}\mathbf{v}$ and $\mathbf{j} = \sum_\alpha q_\alpha \int \mathbf{v} f_\alpha(\mathbf{r}, \mathbf{v}, t)\,\mathrm{d}\mathbf{v}$.

Single-fluid MHD equations:
$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \quad ; \quad \rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \mathbf{j} \times \mathbf{B} - \nabla p \quad ; \quad \frac{\mathrm{d}}{\mathrm{d}t}(p\,\rho^{-\gamma}) = 0$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad ; \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} \quad ; \quad \mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{j}$

Two-fluid MHD equations:
$\frac{\partial n_s}{\partial t} + \nabla\cdot(n_s\mathbf{u}_s) = 0$
$m_s n_s\left[\frac{\partial \mathbf{u}_s}{\partial t} + (\mathbf{u}_s \cdot \nabla)\mathbf{u}_s\right] = q_s n_s(\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) - \nabla \cdot \mathbf{P}_s - m_s n_s \sum_t \nu_{st}(\mathbf{u}_s - \mathbf{u}_t)$

Boltzmann equation: $\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha}\left[\mathbf{E} + \mathbf{v} \times \mathbf{B}\right] \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} = \left(\frac{\partial f_\alpha}{\partial t}\right)_c$

Maxwell-Boltzmann equilibrium distribution:
$f_0(\mathbf{r}, \mathbf{v}) = n(\mathbf{r})\left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-\frac{mv^2}{2k_B T}}$

Kinetic dispersion relation:
$D(k, p) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2}\int_C \frac{\mathrm{d}F_{s0}(v)/\mathrm{d}v}{v - ip/k}\,\mathrm{d}v = 0$